Curvature perturbation at the local extremum of the inflaton's potential

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The spectrum of curvature perturbation generated during inflation is studied in the case the inflation-driving scalar field (inflaton) ϕ crosses over its potential extremum. It is shown that the nondecaying mode of perturbation has a finite value and a proper formula is given. The result is also extended to more general cases where $\ddot{\phi}$ is non-negligible.

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Inflation in the early universe [1] provides a mechanism to generate primordial density fluctuations out of quantum fluctuations of the inflation-driving scalar field, *inflaton* ϕ [2], in addition to explain the large-scale homogeneity and isotropy. In the standard inflation models such as new [3] and chaotic [4] inflation, inflation is driven by the potential energy of the inflaton, $V[\phi]$, as it slowly rolls the potential hill and they predict formation of adiabatic fluctuations with a nearly scale-invariant spectrum. More specifically, the amplitude of the curvature perturbation on comoving hypersurfaces, $\mathcal{R}_c$, on comoving scale $r = 2\pi/k$ is given by the formula [5,6],

$$\mathcal{R}_c(r) \cong \left.\frac{H^2}{2\pi|\dot{\phi}|}\right|_{t_k} \tag{1}$$

$$\cong \left.\frac{3H^3}{2\pi|V'[\phi]|}\right|_{t_k}, \tag{2}$$

where a dot denotes time derivative and H is the Hubble parameter. Here the right-hand-sides should be evaluated at $t = t_k$ when the relevant scale k left the Hubble radius during inflation. In the second approximate equality (2) we have used the equation of motion with the slow-roll approximation,

$$3H\dot{\phi} + V'[\phi] = 0. \tag{3}$$

The reason why (1) gives an almost scale-invariant spectrum is that both H and $\dot{\phi}$ change very slowly during slow-roll inflation.

Recently, however, new classes of inflation models have been proposed in which ϕ is not necessarily slowly rolling during the entire period of inflation [7–9]. Whenever $\ddot{\phi}$ is non-negligible in the full equation of motion of the homogeneous part of ϕ,

$$\ddot{\phi} + 3H\dot{\phi} + V'[\phi] = 0, \tag{4}$$

the slow-roll approximation (3) fails. There are two typical cases where this happens. One is the case $\dot{\phi}$ vanishes. The proper analysis of the amplitude of density fluctuation in this case was recently done in [10], where it was found that the correct amplitude is not given by the formula (1), which is divergent, but by (2). It has been further argued recently [11] that the formula (2) may continue to be a useful approximation even when evolution of ϕ is not described by a slow-roll solution. The other is the case $V'[\phi]$ vanishes, namely, when the inflaton crosses over an extremum of the potential. In this case (2) is divergent and would not be applicable. The purpose of the present paper is first to analyze density fluctuation in this situation and then extend it to the generation of perturbations in more general non-slow-roll inflation models whose potential satisfies several conditions indicated below to allow our analytic treatment.

First, as a specific model we consider the case ϕ overshoots the local maximum of the potential at $\phi = 0$ during the new inflation stage in the chaotic new inflation model [7]. More specifically, we approximate the potential near the extremum as

$$V[\phi] = V_0 - \frac{1}{2}m^2\phi^2, \quad m^2 > 0, \tag{5}$$

and consider the case ϕ crosses over $\phi = 0$ from the positive side to the negative one with a small kinetic energy so that the cosmic expansion rate H is practically constant in the relevant period. We assume $H^2 \gg m^2$ and $V_0 \gg m^2\phi^2$. Then the solution of (4) reads,

$$\phi(t) = \dot{\phi}_* \left[\tfrac{1}{\lambda_+ - \lambda_-} e^{\lambda_+(t-t_*)} - \tfrac{1}{\lambda_+ - \lambda_-} e^{\lambda_-(t-t_*)} \right], \tag{6}$$

$$\dot{\phi}(t) = \dot{\phi}_* \left[\tfrac{\lambda_+}{\lambda_+ - \lambda_-} e^{\lambda_+(t-t_*)} - \tfrac{\lambda_-}{\lambda_+ - \lambda_-} e^{\lambda_-(t-t_*)} \right], \tag{7}$$

$$\lambda_\pm = -\tfrac{3}{2} H \pm \sqrt{\tfrac{9}{4} H^2 + m^2}\,, \tag{8}$$

where we have set $\phi(t_*) = 0$ and $\dot{\phi}(t_*) = \dot{\phi}_*$. Using $\lambda_+ \cong m^2/(3H)$ and $\lambda_- \cong -3H\left(1 + \frac{m^2}{9H^2}\right)$, we find

$$\phi(t) \cong \frac{\dot{\phi}_*}{3H} \left[e^{\frac{m^2}{3H^2} H(t-t_*)} - e^{-3H\left(1+\frac{m^2}{9H^2}\right)(t-t_*)} \right] = \frac{\dot{\phi}_*}{3H} \left[e^{\frac{m^2}{3H^2} H(t-t_*)} - e^{-\frac{m^2}{3H^2} H(t-t_*)} \left(\frac{a(t_*)}{a(t)} \right)^3 \right], \tag{9}$$

$$\begin{aligned} \dot{\phi}(t) &\cong \dot{\phi}_* \left[\frac{m^2}{9H^2} e^{\frac{m^2}{3H^2} H(t-t_*)} + \left(1 - \frac{m^2}{9H^2}\right) e^{-3H\left(1+\frac{m^2}{9H^2}\right)(t-t_*)} \right] \\ &= \dot{\phi}_* \left[\frac{m^2}{9H^2} e^{\frac{m^2}{3H^2} H(t-t_*)} + \left(1 - \frac{m^2}{9H^2}\right) e^{-\frac{m^2}{3H^2} H(t-t_*)} \left(\frac{a(t_*)}{a(t)} \right)^3 \right] \equiv \dot{\phi}_s(t) + \dot{\phi}_r(t), \end{aligned} \tag{10}$$

$$\text{with} \quad \dot{\phi}_s(t) \equiv \dot{\phi}_* \frac{m^2}{9H^2} e^{\frac{m^2}{3H^2} H(t-t_*)}, \quad \dot{\phi}_r(t) \equiv \dot{\phi}_* \left(1 - \frac{m^2}{9H^2}\right) e^{-\frac{m^2}{3H^2} H(t-t_*)} \left(\frac{a(t_*)}{a(t)} \right)^3, \tag{11}$$

near the local maximum of the potential. Here $a(t) \propto e^{Ht}$ denotes the scale factor. As is seen here $\dot{\phi}$ can be decomposed into a slowly varying mode $\dot{\phi}_s(t)$ and a rapidly changing decaying mode $\dot{\phi}_r(t)$ which is approximately proportional to $a^{-3}(t) \propto e^{-3Ht}$.

We now incorporate linear metric perturbations to the spatially flat Robertson-Walker background in the longitudinal gauge,

$$ds^2 = -\left[1 + 2\Psi(\mathbf{x}, t)\right] dt^2 + a^2(t) \left[1 + 2\Phi(\mathbf{x}, t)\right] d\mathbf{x}^2, \tag{12}$$

where we use the notation of [12] for the gauge-invariant perturbation variables [13]. Hereafter all perturbation variables represent Fourier modes like

$$\Phi_{\mathbf{k}} = \int \frac{d^3x}{(2\pi)^{3/2}} \Phi(\mathbf{x}, t) e^{i\mathbf{k}\mathbf{x}}, \tag{13}$$

and we omit the suffix $\mathbf{k}$. We use the following combination of gauge-invariant variables [5,6,10,14,15],

$$Y = X - \frac{\dot{\phi}}{H} \Phi, \tag{14}$$

where

$$X = \delta\phi - \frac{a}{k} \dot{\phi} \sigma_g, \tag{15}$$

is the gauge-invariant scalar field fluctuation with σ_g being the shear of each constant time slice. The latter vanishes on Newtonian slice including the longitudinal gauge.

The above quantity Y is related to the gauge-invariant variable $\mathcal{R}_c$ as

$$\mathcal{R}_c = \Phi - \frac{aH}{k} v = \Phi + \frac{2}{3} \frac{\Phi + H^{-1}\dot{\Phi}}{1 + w} = -\frac{H}{\dot{\phi}} Y, \tag{16}$$

in the present situation where the universe is dominated by the scalar field. Here v is a gauge-invariant velocity perturbation [12] and w denotes the ratio of pressure to energy density.

From the perturbed Einstein equations, we obtain the following equations.

$$\dot{X}\dot{\phi} - \ddot{\phi} X + \dot{\phi}^2 \Phi = \frac{2}{\kappa^2} \frac{k^2}{a^2} \Phi, \tag{17}$$

$$\dot{\Phi} + H\Phi = -\frac{\kappa^2}{2} \dot{\phi} X, \tag{18}$$

where $\kappa^2 = 8\pi G$. From these equations and (14), the equation of motion of Y reads

$$\ddot{Y} + 3H\dot{Y} + \left[\left(\frac{k}{a}\right)^2 + M_{Y\mathrm{eff}}^2\right] Y = 0, \tag{19}$$

with

$$M_{Y\mathrm{eff}}^2 \equiv V''(\phi) + 3\kappa^2\dot{\phi}^2 - \frac{\kappa^4}{2H^2}\dot{\phi}^4 + 2\kappa^2\frac{\dot{\phi}}{H}V'(\phi). \tag{20}$$

This equation has the following exact solution in the long-wavelength limit $k \longrightarrow 0$ [14,16].

$$Y(t) = c_1(k)Y_1(t) + c_2(k)k^3 Y_2(t), \tag{21}$$

$$Y_1(t) = \frac{\dot{\phi}}{H}, \tag{22}$$

$$Y_2(t) = \frac{\dot{\phi}}{H}\int_{t_i}^{t}\frac{H^2}{a^3\dot{\phi}^2}dt, \tag{23}$$

where $c_1(k)$ and $c_2(k)$ are integration constants to be determined by quantum fluctuations generated during inflation, and t_i is some initial time which may be chosen arbitrarily because its effect can be absorbed by redefining $c_1(k)$. We choose t_i so that Y_2 consists only of the decaying mode. Despite its appearance, the decaying mode $Y_2(t)$ is regular even at the epochs $\dot{\phi} = 0$ [14]. In the second term of the right-hand-side of (21), an extra factor k^3 has been introduced so that the scale factor $a(t)$ appears in the rescaling-invariant form of $k/a(t)$, and $c_1(k)$ and $c_2(k)$ have the same dimension [10].

Using (11) we find

$$Y_1(t) = \frac{\dot{\phi}_s(t)}{H} + \frac{\dot{\phi}_r(t)}{H} = \frac{\dot{\phi}_*}{H}\left[\frac{m^2}{9H^2}e^{\frac{m^2}{3H^2}H(t-t_*)} + \left(1 - \frac{m^2}{9H^2}\right)e^{-\frac{m^2}{3H^2}H(t-t_*)}\left(\frac{a(t_*)}{a(t)}\right)^3\right], \tag{24}$$

$$Y_2(t) = -\frac{1}{3\dot{\phi}_s(t)a^3(t)}\left(1 + \frac{2m^2}{9H^2}\right)^{-1} = -\frac{3H^2}{m^2\dot{\phi}_*}\left(1 + \frac{2m^2}{9H^2}\right)^{-1} e^{-\frac{m^2}{3H^2}H(t-t_*)}\frac{1}{a^3(t)}, \tag{25}$$

so that the solution in the long wavelength limit is given by

$$Y(t) = c_1(k)\left(\frac{\dot{\phi}_s(t)}{H} + \frac{\dot{\phi}_r(t)}{H}\right) - c_2(k)\frac{H^3}{3\dot{\phi}_s(t)}\left(1 + \frac{2m^2}{9H^2}\right)^{-1}\left(\frac{k}{a(t)H}\right)^3 \tag{26}$$

We can also incorporate the finite wavenumber correction using the iterative expression [10,14] as

$$Y = c_1(k)Y_1 + c_2(k)k^3Y_2 + k^2Y_1\int aY_2Y\,dt - k^2Y_2\int aY_1Y\,dt. \tag{27}$$

In the present case we find the leading finite-wavenumber correction is proportional to $(k/a(t))^2$. As a result we obtain

$$\begin{aligned} Y(t) = {} & c_1(k)\frac{\dot{\phi}_s(t)}{H} + c_1(k)\frac{\dot{\phi}_s(t)}{2H}\left(1 - \frac{2m^2}{3H^2}\right)\left(\frac{k}{a(t)H}\right)^2 \\ & + c_1(k)\frac{\dot{\phi}_r(t)}{H} - c_2(k)\frac{H^3}{3m^2\dot{\phi}_s(t)}\left(1 - \frac{2m^2}{9H^2}\right)\left(\frac{k}{a(t)H}\right)^3 + \cdots, \end{aligned} \tag{28}$$

to the lowest-order in m^2/H^2.

Next we consider evolution of Y in its short-wavelength regime in order to set the initial condition out of quantum fluctuations. Since we are primarily interested in k-mode that leaves the horizon when the inflaton crosses over its potential extremum at $t = t_*$, we only need to consider the evolution of Y when $m^2|t - t_*|/3H \ll 1$ holds during inflation. Thus in this regime (19) becomes

$$\ddot{Y} + 3H\dot{Y} + \left(\frac{k}{a}\right)^2 Y = 0. \tag{29}$$

Under the condition $|\dot{H}/H^2| \ll 1$ the solution of this equation satisfying the normalization condition,

$$Y\dot{Y}^* - \dot{Y}Y^* = \frac{i}{a^3}, \tag{30}$$

is approximately given by

$$Y = \frac{iH}{\sqrt{2k^3}} \left[\alpha_{\mathbf{k}}(1 + ik\eta)e^{-ik\eta} - \beta_{\mathbf{k}}(1 - ik\eta)e^{ik\eta} \right], \tag{31}$$

where $\eta = -1/(Ha)$ is a conformal time, and $\alpha_{\mathbf{k}}$ and $\beta_{\mathbf{k}}$ are constants which satisfy $|\alpha_{\mathbf{k}}|^2 - |\beta_{\mathbf{k}}|^2 = 1$. We shall choose $(\alpha_{\mathbf{k}}, \beta_{\mathbf{k}}) = (1, 0)$ so that the vacuum reduces to the one in Minkowski spacetime at the short-wavelength limit $(-k\eta \to \infty)$. We therefore obtain

$$Y = \frac{iH}{\sqrt{2k^3}}(1 + ik\eta)e^{-ik\eta} = \frac{iH}{\sqrt{2k^3}} \left[1 + \frac{1}{2}\left(\frac{k}{a(t)H}\right)^2 + \frac{i}{3}\left(\frac{k}{a(t)H}\right)^3 + ... \right], \tag{32}$$

where the latter expansion is useful after k-mode has gone out of the horizon in the period $t < t_* + 3H^2/m^2$.

From (32) and (28), we find that under the condition $m^2 \ll H^2$ we can match the first and the second terms of (28) with those of (32) and the third and the fourth terms of (28) with the third term of (32). The former constitutes the growing (or more properly, nondecreasing) mode and the latter corresponds to the decaying mode. As a result we find

$$c_1(k) = \frac{iH^2}{\sqrt{2k^3}\dot{\phi}_s(t_k)} = \frac{9iH^4}{\sqrt{2k^3}m^2\dot{\phi}_*} e^{-\frac{m^2}{3H^2}H(t_k - t_*)}, \tag{33}$$

$$c_2(k) = \frac{\dot{\phi}_s(t_k)}{\sqrt{2k^3}H^2}\left(1 + \frac{2m^2}{9H^2}\right)\left[1 + i\frac{27H^2}{m^2}\left(1 - \frac{m^2}{9H^2}\right)e^{-\frac{2m^2}{3H^2}H(t_k - t_*)}\left(\frac{a(t_*)H}{k}\right)^3\right]. \tag{34}$$

This means that the nondecaying mode of the curvature perturbation on the comoving horizon scale at $t = t_*$, r_*, is given by,

$$\mathcal{R}_c(r_*) = \left[|c_1(k)|^2 \frac{4\pi k^3}{(2\pi)^3}\right]^{1/2} = \frac{9H^4}{2\pi m^2|\dot{\phi}_*|}. \tag{35}$$

Apparently it is enhanced by a factor $9H^2/m^2$ compared with (1). This does not mean, however, that we find a severe deviation from a scale-invariant spectrum. On a more general comoving scale $r = r(t_k) = 2\pi/k$ leaving the Hubble radius during new inflation, we find from the first equatlity of (33),

$$\mathcal{R}_c(r) = \frac{H^2}{2\pi|\dot{\phi}_s(t_k)|}, \tag{36}$$

which is practically scale-invariant for $m^2 \ll H^2$. The deviation from the scale-invariant spectrum is manifest only in the decaying mode $c_2(k)$, as is seen in the second term of (34).

Thus we can obtain the correct expression for the nondecreasing mode of the curvature perturbation if we replace $\dot{\phi}$ in (1) with its nondecaying mode only. This was also the case when $\dot{\phi}$ vanishes during inflation which was studied in [10]. Figure 1 depicts the actual spectrum of curvature perturbation together with the conventional slow-roll formula (1) for $H^2/m^2 = 30$. Using (1) would seriously underestimate the actual amplitude in the regime ϕ is not slowly rolling.

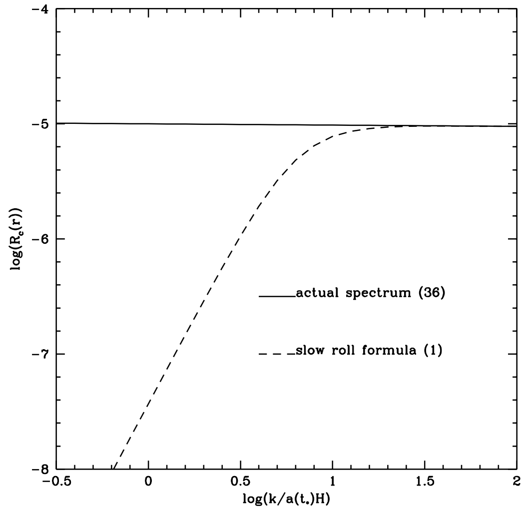

FIG. 1. Actual spectrum of the curvature perturbation (solid line) and the result of slow-roll formula (broken line) for $H^2/m^2 = 30$.

Next we consider the applicability of this approach to a more general potential $V[\phi]$. As is clear from the above analysis our method is applicable if the solution of $\dot{\phi}$ is separable to a sum of a slowly varying mode and a decaying mode approximately proportional to $a^{-3}(t)$ at least for a few expansion time scales after the k-mode under consideration leaves the Hubble radius at $t = t_k$. Let us expand $V[\phi]$ around $\phi = \phi(t_k) \equiv \phi_k$ up to second order:

$$V[\phi] = V[\phi_k] + V'[\phi_k](\phi - \phi_k) + \frac{1}{2}V''[\phi_k](\phi - \phi_k)^2 + \cdots \equiv V_k + V_k'(\phi - \phi_k) + \frac{1}{2}V_k''(\phi - \phi_k)^2 + \cdots. \tag{37}$$

Then a solution of the equation of motion, $\ddot{\phi} + 3H_k\dot{\phi} + V_k' + V_k''(\phi - \phi_k) = 0$, is given by a linear combination of $e^{\Lambda_+(t-t_k)}$ and $e^{\Lambda_-(t-t_k)}$, where $\Lambda_\pm$ are solutions of $\Lambda^2 + 3H_k\Lambda + V_k'' = 0$. Here a subscript k denotes the value of each quantity at the time t_k.

In order that the solution of $\dot{\phi}$ is separable to a slowly changing mode and a decaying mode, we require $|V_k''| \ll H_k^2$. Then we find, to the lowest order in $|V_k''|/H_k^2$,

$$\dot{\phi}(t) = \left[\frac{V_k''}{9H_k^2}\dot{\phi}_k + \left(1 - \frac{V_k''}{9H_k^2}\right)\frac{V_k'}{3H_k}\right] e^{\frac{V_k''}{3H_k^2}H_k(t-t_k)} + \left(1 - \frac{V_k''}{9H_k^2}\right)\left(\dot{\phi}_k - \frac{V_k'}{3H_k}\right) e^{-3\left(1+\frac{V_k''}{9H_k^2}\right)H_k(t-t_k)}. \tag{38}$$

The first and the second terms of the above solution correspond to $\dot{\phi}_s(t)$ and $\dot{\phi}_r(t)$ in (10) of the previous analysis, respectively. If we apply the same matching method as before, we obtain

$$\mathcal{R}_c(r) = \frac{H_k^2}{2\pi}\left|\frac{V_k''}{9H_k^2}\dot{\phi}_k + \left(1 - \frac{V_k''}{9H_k^2}\right)\frac{V_k'}{3H_k}\right|^{-1}, \tag{39}$$

instead of (36) for the nondecaying mode of the curvature perturbation on the comoving horizon scale at $t = t_k$. In order that (39) gives the correct expression for the curvature perturbation, the expansion (37) should remain valid at least for a few expansion time scales after k-mode has crossed the horizon at $t = t_k$. That is, we require

$$|V_k'||\dot{\phi}_k|H_k^{-1} < V_k, \qquad |V_k''|(\dot{\phi}_k H_k^{-1})^2 < V_k. \tag{40}$$

Since the k-mode leaves the horizon during inflation, we have $\dot{\phi}_k^2 < V_k$. Then together with the assumption $|V_k''| \ll H_k^2$, the second inequality is trivially satisfied. Furthermore, since V_k'/H_k is of the same order of $\dot{\phi}$ in the slow-roll case, we find $|V_k'||\dot{\phi}_k|H_k^{-1} \lesssim \dot{\phi}_k^2 < V_k$ is also automatically satisfied. Thus the only nontrivial condition for our analysis to be valid is $|V_k''| \ll H_k^2$ apart from the condition for accelerated expansion $\dot{\phi}_k^2 < V_k$.

In summary, we have analytically studied generation of curvature perturbation in the case the slow-roll equation of motion does not hold during inflation, by matching the quantum fluctuation (32) with the long-wave exact solution (26). For this matching to be possible it is essential that the solution of $\dot{\phi}$ can be described by a sum of a slowly changing mode and a decaying mode approximately proportional to $a^{-3}(t)$, which requires $|V_k''| \ll H_k^2$. Models with more complicated potentials that violate this condition must be studied numerically [11,17].

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